

MTH250 - CALCULUS

Lecture No. 23

DR. ABDUL WAHAB

Assistant Professor,
Department of Mathematics, CIIT, Pakistan.

© **VIRTUAL CAMPUS**

COMSATS Institute of Information Technology
CIIT, Virtual Campus Secretariat 402, Street 34, I-8/2 Islamabad-Pakistan.
Ph:+92 51 9101237-39 | info@vcomsats.edu.pk | vcomsats.edu.pk

These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by H. Anton, I Bivens and S. Davis, John Wiley & Sons, 10th edition, 2012.

Contents

Contents	i
23 Multivariable fuctions I	1
23.1 Multivariable functions	1
23.2 Limits along smooth curves	3
23.3 Limits of multivariable functions	5
23.4 Continuity of multivariable functions	6
23.5 Partial differentiation	8
23.5.1 Partial differentiation as rate of change and slope of the tangent	9

LECTURE 23

Multivariable functions I

23.1 Multivariable functions

Definition 23.1. A function f of two variables, x , and y , is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D in the xy -plane.

In general, a function f of n variables, $x_1, x_2, \dots, x_n$ is a rule that assigns a unique real number $f(x_1, x_2, \dots, x_n)$ to each point $(x_1, x_2, \dots, x_n)$ in some set D in the $\mathbb{R}^n$. The set D is called its domain and the set $\{f(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots, x_n) \in D\}$ is called its range.

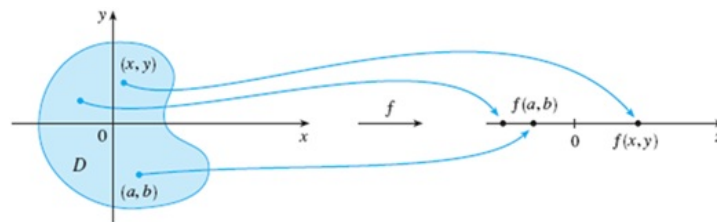


Figure 23.1: Multivariable function.

Example 23.1. Let $f(x, y) = \sqrt{y+1} + \ln(x^2 - y)$. Find $f(e, 0)$ and sketch the natural domain of f .

Solution. By substitution,

$$f(e, 0) = \sqrt{0+1} + \ln(e^2 - 0) = 1 + \ln e^2 = 1 + 2\ln e = 3.$$

To find the natural domain of f , we note that $\sqrt{y+1}$ is defined only when $y \geq -1$, while $\ln(x^2 - y)$ is defined only when $0 < x^2 - y$ or $y < x^2$. Thus, the natural domain of f consists of all points in the xy -plane for which $-1 \leq y < x^2$. To sketch the natural domain, we first sketch the parabola $y = x^2$ as a *dashed* curve and the line $y = -1$ as a solid curve. The natural domain of f is then the region lying above or on the line $y = -1$ and below the parabola $y = x^2$; see Figure 23.2. $\square$

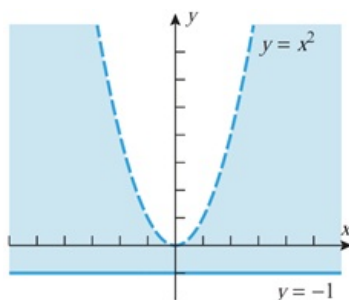


Figure 23.2: Natural domain of f , the solid boundary line is included in the domain, while the dashed boundary is not included.

Definition 23.2. If f is a function of two variables with domain D , then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D .

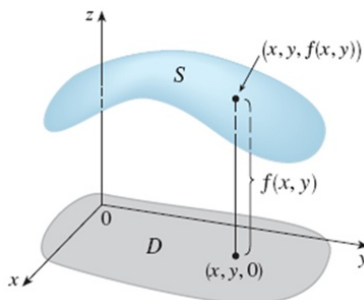


Figure 23.3: Graph of a multivariable function with domain D .

Example 23.2. The graph of the function $f(x, y) = -\sqrt{x^2 + y^2}$ is as shown in Figure 23.4.

Definition 23.3 (Level curves).

The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

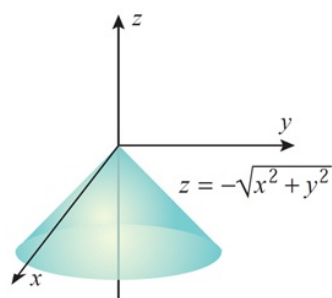


Figure 23.4:

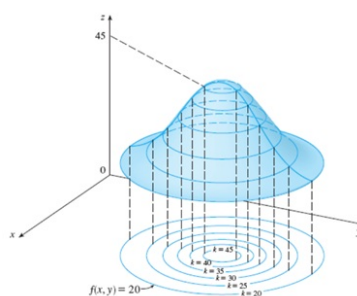


Figure 23.5: Level curves.

23.2 Limits along smooth curves

For a function of one variable there are two one-sided limits at a point x_0 , namely

$$\lim_{x \rightarrow x_0^+} f(x) \quad \text{and} \quad \lim_{x \rightarrow x_0^-} f(x)$$

There are infinite different curves along which one point can approach another.

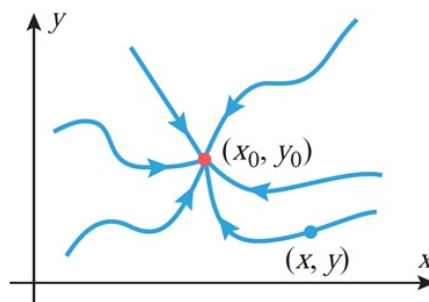


Figure 23.6: Level curves.

Definition 23.4 (Limit along smooth curves). *If C is a smooth parametric curve in two and three dimension that is represented by $x = x(t)$, $y = y(t)$ or $x = x(t)$, $y = y(t)$, $z = z(t)$. Then,*

$$\lim_{\substack{(x,y) \rightarrow (x_0,y_0) \\ \text{(along } C)}} f(x, y) = \lim_{t \rightarrow t_0} f(x(t), y(t))$$

$$\lim_{\substack{(x,y,z) \rightarrow (x_0,y_0,z_0) \\ \text{(along } C)}} f(x, y, z) = \lim_{t \rightarrow t_0} f(x(t), y(t), z(t)).$$

Remark 23.5. *In these formulae the limit of the function of t must be treated as one sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C .*

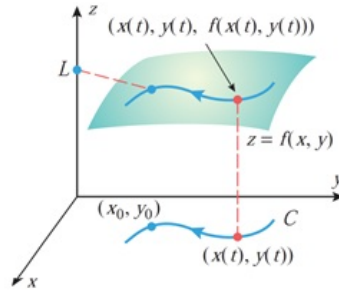


Figure 23.7: Limit along a smooth curve C .

Example 23.3. *Find the limit of $f(x, y)$ as $(x, y) \rightarrow (0, 0)$ along $y = -x$ and $y = x^2$.*

Solution. The line $y = -x$ has parametric equations $x = t$, $y = -t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{(along } y=-x)}} f(x, y) = \lim_{t \rightarrow 0} f(t, -t) = \lim_{t \rightarrow 0} \frac{t^2}{2t^2} = \lim_{t \rightarrow 0} \frac{1}{2} = \frac{1}{2}.$$

The parabola $y = x^2$ has parametric equations $x = t$, $y = t^2$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{(along } y=x^2)}} f(x, y) = \lim_{t \rightarrow 0} f(t, t^2) = \lim_{t \rightarrow 0} -\frac{t^3}{t^2 + t^4} = \lim_{t \rightarrow 0} -\frac{t}{1 + t^2} = 0.$$

□

Definition 23.6 (Open and closed sets).

Let $(x_0, y_0) \in \mathbb{R}^2$ and $\delta > 0$. The set of all point that are enclosed by the circle centered at (x_0, y_0) and radius δ , but do not lie on the circle, is called the open disk of radius δ and center (x_0, y_0) . The set of points that lie on the circle together with those enclosed by the circle is called the closed disk.

Analogously, if S is a sphere centered at (x_0, y_0, z_0) and has positive radius δ , then the set of points that enclosed by the sphere, but do not lie on the sphere, is called the open ball. The set of points that lie on the sphere together with those enclosed by the sphere is called the closed ball.

If D is a set in two dimensions, then (x_0, y_0) called an interior point of D if there is some open disk centered at (x_0, y_0) that contains only points of D , and is called a boundary point if every open disk centered at (x_0, y_0) contains both points in D and points not in D .

A set D is called closed if it contains all of its boundary points and open if it contains none of its boundary points. $\mathbb{R}^2$ and $\mathbb{R}^3$ are both closed and open.

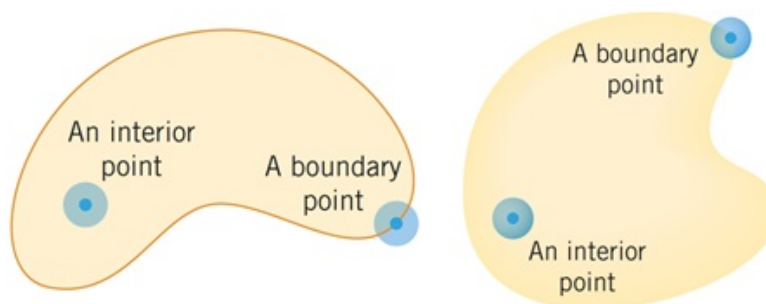


Figure 23.8: Left: Closed set, Right: An open set.

23.3 Limits of multivariable functions

Definition 23.7 (Limits of Multivariable functions: Formal).

Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0) , except possibly at (x_0, y_0) . We will write

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x,y)$ satisfies

$$|f(x,y) - L| < \epsilon$$

whenever the distance between (x,y) and (x_0, y_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Example 23.4. Find $\lim_{(x,y) \rightarrow (1,4)} [5x^3y^2 - 9]$.

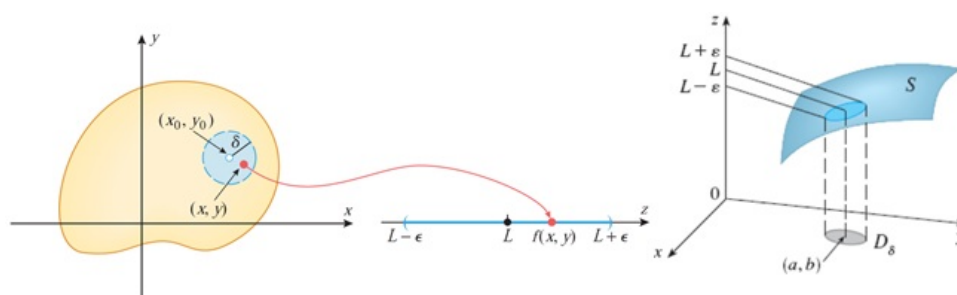


Figure 23.9: Formal definition of limit.

Solution.

$$\begin{aligned}
 \lim_{(x,y) \rightarrow (1,4)} [5x^3y^2 - 9] &= \lim_{(x,y) \rightarrow (1,4)} [5x^3y^2] - \lim_{(x,y) \rightarrow (1,4)} [9] \\
 &= 5 \left[\lim_{(x,y) \rightarrow (1,4)} x \right]^3 \left[\lim_{(x,y) \rightarrow (1,4)} y \right]^2 - 9 \\
 &= 5(1)^3(4)^2 - 9 = 71.
 \end{aligned}$$

□

Theorem 23.8. If $f(x, y) \rightarrow L$ as $(x, y) \rightarrow (x_0, y_0)$, then $f(x, y) \rightarrow L$ as $(x, y) \rightarrow (x_0, y_0)$ along any smooth curve.

If the limit of $f(x, y)$ fails to exist as $(x, y) \rightarrow (x_0, y_0)$ along some smooth curve or if $f(x, y)$ has different limits as $(x, y) \rightarrow (x_0, y_0)$ along two different smooth curves, then the limit of $f(x, y)$ does not exist as $(x, y) \rightarrow (x_0, y_0)$.

Example 23.5. The limit

$$\lim_{(x,y) \rightarrow (0,0)} \left(-\frac{xy}{x^2 + y^2} \right)$$

does not exist because

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } x=0}} \left(-\frac{xy}{x^2 + y^2} \right) = 0 \quad \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x}} \left(-\frac{xy}{x^2 + y^2} \right) = -\frac{1}{2}.$$

23.4 Continuity of multivariable functions

Definition 23.9. A function $f(x, y)$ is said to be continuous at (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0).$$

In addition, if f is continuous at every point in an open set D , then we say that f is continuous on D , and if f is continuous at every point in the xy -plane, then we say that f is continuous everywhere.

Theorem 23.10. *If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0 , then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0) .*

If $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$, then the composition $f(x, y) = g(h(x, y))$ is continuous at (x_0, y_0) .

If $f(x, y)$ is continuous at (x_0, y_0) , and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0 .

Remark 23.11 (Recognizing a continuous function).

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

Example 23.6. Evaluate $\lim_{(x,y) \rightarrow (-1,2)} \left(\frac{xy}{x^2 + y^2} \right)$.

Solution. Since $f(x, y) = \frac{xy}{x^2 + y^2}$ is continuous at $(-1, 2)$, it follows from the definition of continuity for function of two variables that

$$\lim_{(x,y) \rightarrow (-1,2)} \left(\frac{xy}{x^2 + y^2} \right) = \frac{(-1)(2)}{(-1)^2 + (2)^2} = -\frac{2}{5}.$$

□

Example 23.7. The function $f(x, y) = \frac{x^2 y^2}{1 - xy}$ is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus $f(x, y)$ is continuous everywhere except on the hyperbola $xy = 1$.

Definition 23.12 (Limits in 3D).

Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0, z_0) , except possibly at (x_0, y_0, z_0) . We will write

$$\lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x, y, z) = L$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y, z)$ satisfies

$$|f(x, y, z) - L| < \epsilon$$

whenever the distance between (x, y, z) and (x_0, y_0, z_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta.$$

23.5 Partial differentiation

If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f , then the partial derivative of f with respect to x at (x_0, y_0) [also called the partial derivative of z with respect to x at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0)] \Big|_{x=x_0} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

Similarly, the partial derivative of f with respect to y at (x_0, y_0) [also called the partial derivative of z with respect to y at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)] \Big|_{y=y_0} = \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Example 23.8. Find $f_x(1, 3)$ and $f_y(1, 3)$ for the function $f(x, y) = 2x^3y^2 + 2y + 4x$.

Solution. Since

$$f_x(x, 3) = \frac{d}{dx}[f(x, 3)] = \frac{d}{dx}[18x^3 + 4x + 6] = 54x^2 + 4,$$

we have $f_x(1, 3) = 54 + 4 = 58$.

Also, since,

$$f_y(1, y) = \frac{d}{dy}[f(1, y)] = \frac{d}{dy}[2y^2 + 2y + 4] = 4y + 2,$$

we have $f_y(1, 3) = 4(3) + 2 = 14$. □

If $z = f(x, y)$, then the partial derivatives f_x and f_y are also denoted by the symbols,

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and} \quad \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of $z = f(x, y)$ at a point (x_0, y_0) are

$$\frac{\partial f}{\partial x} \Big|_{x=x_0, y=y_0}, \quad \frac{\partial z}{\partial x} \Big|_{(x_0, y_0)}, \quad \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0).$$

Example 23.9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$.

Solution.

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial x}(x^4) \\ &= x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot (4x^3) = x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3). \end{aligned}$$

$$\begin{aligned}
 \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial y}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial y}(x^4) \\
 &= x^4 \cos(xy^3) \cdot 3xy^2 + 0 = 3x^5 y^2 \cos(xy^3).
 \end{aligned}$$

□

23.5.1 Partial differentiation as rate of change and slope of the tangent

Recall that for $y = f(x)$, the value $f'(x_0)$ is interpreted either as the rate of change of y with respect to x at x_0 , or the slope of the tangent line at $x = x_0$. The partial derivatives have an analogous interpretation.

Suppose C_1 is the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$. Then, $f_x(x, y = y_0)$ can be interpreted as the rate of change of z with respect to x along the curve C_1 . In particular, $f_x(x_0, y_0)$ is the rate of change of z with respect to x along the curve C_1 at point (x_0, y_0) .

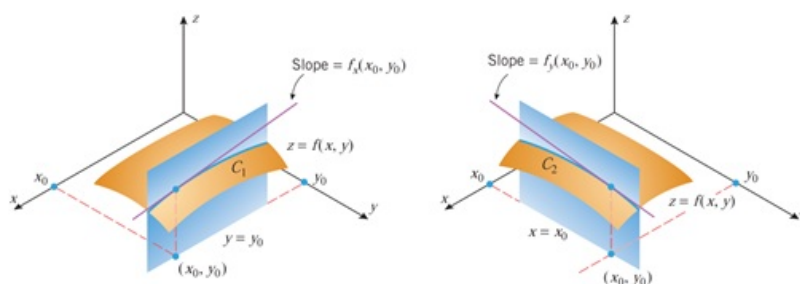


Figure 23.10: Partial derivative as a slope of the tangent. C_1 and its tangent (left). C_2 and its tangent (right).

Example 23.10. Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y -direction at the points $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$.

Solution. The point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ lies on the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, and the point $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$ lies on the lower hemisphere $z = -\sqrt{1 - x^2 - y^2}$. We could find the slopes by differentiating each expression for z separately with respect to y and then evaluating the derivatives at $x = \frac{2}{3}$ and $y = \frac{1}{3}$. However, it is more efficient to differentiate the given equation

$$x^2 + y^2 + z^2 = 1$$

implicitly with respect to y , since this will give us both slopes with one differentiation. To perform the implicit differentiation, we view z as a function of x and y and

differentiate both sides with respect to y , taking x to be fixed. The computations are as follows:

$$\frac{\partial}{\partial y}[x^2 + y^2 + z^2] = \frac{\partial}{\partial y}[1] \implies 2y + 2z \frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y} = -\frac{y}{z}$$

Substituting the y - and z -coordinates of the points $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$ in this expression, we find that the slope at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ is $-\frac{1}{2}$ and the slope at $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$ is $\frac{1}{2}$. $\square$

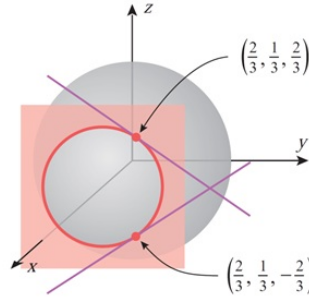


Figure 23.11: Partial derivative as a slope of the tangent. C_1 and its tangent (left). C_2 and its tangent (right).

Example 23.11. Find the second order partial derivatives of $f(x, y) = x^2 y^3 + x^4 y$.

Solution. We have

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y, \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4.$$

so that

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (2xy^3 + 4x^3y) = 2y^3 + 12x^2y. \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (3x^2y^2 + x^4) = 6x^2y. \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (3x^2y^2 + x^4) = 6x^2y + 4x^3. \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (2xy^3 + 4x^3y) = 6x^2y + 4x^3. \end{aligned}$$

$\square$

Notice that, from the above example, that $f_{xy} = f_{yx}$. This is not a mere coincidence and in general we have the following theorem.

Theorem 23.13. Let f be a function of two variables. If f_{xy} , and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on the disk.